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CAUCHY PROBLEM OF NONLOCAL FRACTIONAL CONFORMABLE INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. In this work, we consider the study of the existence of mild solutions of a Cauchy problem governed by a conformable fractional derivative of positive and inferior than one order and an infinitesimal generator of a cosine family on a Banach space. The idea of our main result is based on the use of the Schaefer's fixed point theorem combined with the cosine family of linear operators.

Keywords: Fractional differential equations, Integro-ordinary differential equations, Cosine family of linear operators, Conformable fractional derivatives, Nonlocal conditions.

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1 Introduction

In the works Atraoui & Bouaouid (2021); Bouaouid et al. (2019) the authors proved some results concerning the existence of mild solutions for the following Cauchy problem:

$$\begin{cases} \frac{d^\alpha}{dt^\alpha}\left(\frac{d^\alpha x(t)}{dt^\alpha}\right) = Ax(t) + f(t, x(t)), \ \ t \in [0, \tau], \ \ 0 < \alpha \leq 1, \\ \\ x(0) = x_0 + g(x), \ \ \frac{d^\alpha x(0)}{dt^\alpha} = x_1 + h(x), \end{cases} \tag{1}$$

where τ is a positive real number, $\frac{d^\alpha(.)}{dt^\alpha}$ presents the conformable fractional derivative of order $\alpha \in]0, 1]$ (Khalil et al. (2014)), A is the infinitesimal generator of a cosine family $(\{C(t), S(t)\})_{t\in\mathbb{R}}$ on a Banach space X (Travis & Webb (1978)); and the elements x_0 and x_1 are two fixed vectors in X. The expressions $x(0) = x_0 + g(x)$ and $\frac{d^\alpha x(0)}{dt^\alpha} = x_1 + h(x)$ mean the so-called nonlocal conditions Byszewski (1991). The functions $f : [0, \tau]\times X \longrightarrow X$ and $g, h : \mathcal{C} \longrightarrow X$ satisfied some assumptions, where $\mathcal{C}$ is the Banach space of continuous functions from $[0, \tau]$ into X equipped with the norm $\mid x \mid_c = \sup_{t\in[0,\tau]} \parallel x(t) \parallel$.

In fact, fractional derivatives and nonlocal conditions prove that they are best approaches to model dynamical systems with memory in various fields of scientific such as physics, engineering, optimization, data science, biology, finance, chemistry, and so on Guliyev (2021); Laghrib (2020); Nachaoui (2020); Nachaoui et al. (2021).

The present contribution is a continuation of the works Atraoui & Bouaouid (2021); Bouaouid et al. (2019), in order to consider the same Cauchy problem given in (1) with an integral term. Precisely, we are interested with the study of the nonlocal fractional conformable integro-differential equations of the following form:

$$\begin{cases} \frac{d^\alpha}{dt^\alpha}(\frac{d^\alpha x(t)}{dt^\alpha}) = Ax(t) + f(t, x(t)) + \int_0^t a(t-\sigma)\varphi(\sigma, x(\sigma))d\sigma, \quad t \in [0, \tau], \ \ 0 < \alpha \leq 1, \\ \\ x(0) = x_0 + g(x), \ \ \frac{d^\alpha x(0)}{dt^\alpha} = x_1 + h(x), \end{cases} \tag{2}$$

where $a : [0, \tau] \longrightarrow \mathbb{R}$, $\varphi : [0, \tau] \times X \longrightarrow X$ are two functions satisfied some assumptions and X is supposed a real vector space.

The rest of this work is organized as follows. In section 2, we recall some preliminary facts on the conformable fractional calculus and theory of cosine family of linear operators. Section 3 is devoted to prove the main result based the Schaefer fixed point theorem combined with the following Duhamel formula

$$x(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})[f(s, x(s)) + k(s, x(s))]ds, \tag{3}$$

where $k(s, x(s))$ is the convolution operator given as follows:

$$k(s, x(s)) = \int_0^s a(s-\sigma)\varphi(\sigma, x(\sigma))d\sigma. \tag{4}$$

2 Preliminaries

We recall some preliminary facts on the conformable fractional calculus.

Definition 1. *(Khalil et al. (2014)) The conformable fractional derivative of order $\alpha \in (0, 1]$ for a function $x(.)$ is defined by*

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\varepsilon \longrightarrow 0} \frac{x(t + \varepsilon t^{1-\alpha}) - x(t)}{\varepsilon}, \ \ t > 0,$$

$$\frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t \longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha},$$

provided that the limits exist.

The fractional integral $\mathrm{I}^\alpha(.)$ associated with the conformable fractional derivative is defined by

$$\mathrm{I}^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s)ds.$$

Theorem 1. *(Khalil et al. (2014)) If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$\frac{d^\alpha(I^\alpha(x)(t))}{dt^\alpha} = x(t).$$

Definition 2. *(Abdeljawad (2015)) The fractional Laplace transform of order $\alpha \in (0, 1]$ for a function $x(.)$ is defined by*

$$\mathcal{L}_\alpha(x(t))(\lambda) = \int_0^{+\infty} t^{\alpha-1} e^{-\lambda \frac{t^\alpha}{\alpha}} x(t)dt, \ \ \lambda > 0.$$

The following proposition gives us the actions of the fractional integral and the fraction Laplace transform on the conformable fractional derivative, respectively.

Proposition 1. *(Abdeljawad (2015)) For a differentiable function $x(.)$, we have the following results*

$$I^{\alpha}\Big(\frac{d^{\alpha}x(.)}{dt^{\alpha}}\Big)(t) = x(t) - x(0),$$

$$\mathcal{L}_{\alpha}\Big(\frac{d^{\alpha}x(t)}{dt^{\alpha}}\Big)(\lambda) = \lambda\mathcal{L}_{\alpha}(x(t))(\lambda) - x(0).$$

According to Bouaouid et al. (2018), we have the following remark.

Remark 1. *For two functions $x(.)$ and $y(.)$, we have*

$$\mathcal{L}_{\alpha}\big(x\big(\frac{t^{\alpha}}{\alpha}\big)\big)(\lambda) = \mathcal{L}_{1}(x(t))(\lambda),$$

$$\mathcal{L}_{\alpha}\Big(\int_0^t s^{\alpha-1}x\big(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\big)y(s)ds\Big)(\lambda) = \mathcal{L}_{1}(x(t))(\lambda)\mathcal{L}_{\alpha}(y(t))(\lambda).$$

Now, we present some definitions and results concerning the cosine family of linear operators Travis & Webb (1978).

Definition 3. *(Travis & Webb (1978)) A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on a Banach space X is called a strongly continuous cosine family if and only if*

1. *$C(0) = I$.*
2. *$C(s+t) + C(s-t) = 2C(s)C(t)$ for all $t, s \in \mathbb{R}$.*
3. *The function $t \longmapsto C(t)x$ is continuous for each fixed $x \in X$.*

We also define the sine family $(S(t))_{t\in\mathbb{R}}$ associated with the cosine family $(C(t))_{t\in\mathbb{R}}$ as follows

$$S(t)x = \int_0^t C(s)xds, \;\; x \in X.$$

The infinitesimal generator A of a strongly continuous cosine family $(\{C(t), S(t)\})_{t\in\mathbb{R}}$ on X is defined by

$$D(A) = \{x \in X, \; t \longmapsto C(t)x \text{ is a twice continuously differentiable function}\}$$

and

$$Ax = \frac{d^2C(0)x}{dt^2}, \;\; x \in D(A).$$

Proposition 2. *(Travis & Webb (1978)) The following assertions are true.*

1. *There exist two constants $K \geq 1$ and $\omega \geq 0$ such that*

$$\mid S(t) - S(s) \mid_{\mathcal{L}(X)} \leq K \mid \int_s^t \exp(\omega \mid r \mid) \mid dr \; for\; all\, t, s \in \mathbb{R},$$

where $\mid . \mid_{\mathcal{L}(X)}$ denote the norm in the space $\mathcal{L}(X)$ of bounded operators defined from X into itself, which we will be denoted it only by $\mid . \mid$.

2. *For $\lambda \in \mathbb{C}$ such that $Re(\lambda) > \omega$, we have*

$$\lambda^2 \in \rho(A), \; (\rho(A) \textit{ is the resolvent set of } A),$$

$$\lambda(\lambda^2 I - A)^{-1}x = \int_0^{+\infty} e^{-\lambda t}C(t)xdt \; for\; all\, x \in X,$$

$$(\lambda^2 I - A)^{-1}x = \int_0^{+\infty} e^{-\lambda t}S(t)xdt \; for\; all\, x \in X.$$

3 Main results

We first state the following lemma.

Lemma 1. *(Bouaouid et al. (2019)) Every solution of Cauchy problem* (2) *satisfies the following integral equation*

$$x(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})[f(s, x(s)) + k(s, x(s))]ds,$$

where $k(s, x(s))$ *is the convolution operator given as follows:*

$$k(s, x(s)) = \int_0^s a(s-\sigma)\varphi(\sigma, x(\sigma))d\sigma.$$

Definition 4. *(Bouaouid et al. (2019)) A function* $x \in \mathcal{C}$ *is called a mild solution of the Cauchy problem* (2) *if*

$$x(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})[f(s, x(s)) + k(s, x(s))]ds.$$

To obtain the existence of mild solutions, we will need the following assumptions:

(H_1) The function $f(.,.) : [0,\tau] \times X \longrightarrow X$ is continuous and there exists a function $\mu \in L^\infty([0,\tau], \mathbb{R}^+)$ such that $\| f(t,x) \| \leq \mu(t)$ for all $t \in [0,\tau]$ and $x \in X$.

(H_2) The function $k(.,.) : [0,\tau] \times X \longrightarrow X$ is continuous and there exists a function $\nu \in L^\infty([0,\tau], \mathbb{R}^+)$ such that $\| k(t,x) \| \leq \nu(t)$ for all $t \in [0,\tau]$ and $x \in X$.

(H_3) The function $g : \mathcal{C} \longrightarrow X$ is continuous and compact.

(H_4) There exist positive constants a and b such that $\| g(x) \| \leq a \mid x \mid_c + b$ for all $x \in \mathcal{C}$.

(H_5) The function $h : \mathcal{C} \longrightarrow X$ is continuous and compact.

(H_6) There exist positive constants c and d such that $\| h(x) \| \leq c \mid x \mid_c + d$ for all $x \in \mathcal{C}$.

(H_7) The family $(\{C(t), S(t)\})_{t\in\mathbb{R}}$ is uniformly continuous, that is $\lim\limits_{t\longrightarrow s} |C(t) - C(s)| = 0$ and $\lim\limits_{t\longrightarrow s} |S(t) - C(s)| = 0$.

(H_8) The set $\left\{ \int_0^{t-h} S(t-s)xds \right\}$ is relatively compact in X for $0 < h \leq t$ and x is a fixed element in X.

Theorem 2. *Assume that* $(H_1) - (H_8)$ *hold, then Cauchy problem* (2) *has at least one mild solution provided that*

$$a \sup_{t\in[0,\tau]} |C(\frac{t^\alpha}{\alpha})| + c \sup_{t\in[0,\tau]} |S(\frac{t^\alpha}{\alpha})| < 1.$$

Proof. In order to use the Schaefer fixed point theorem, we define the operator $\Gamma : \mathcal{C} \longrightarrow \mathcal{C}$ as follows

$$\Gamma(x)(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] + \int_0^t s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})[f(s, x(s)) + k(s, x(s))]ds.$$

We also consider the following notations:

$$B_r = \{x \in \mathcal{C}, \ \mid x \mid_c \leq r \text{ with } r > 0\},$$

$$\Omega(\Gamma) = \{x \in \mathcal{C}, \ x = \lambda\Gamma(x) \text{ with } \lambda \in]0,1[\}.$$

The proof will be given in three steps:

Step 1: Prove that Γ is continuous.

Let $(x_n) \subset \mathcal{C}$ such that $x_n \longrightarrow x$ in $\mathcal{C}$. We have

$$\begin{aligned}\Gamma(x_n)(t) - \Gamma(x)(t) &= C\big(\frac{t^\alpha}{\alpha}\big)[g(x_n) - g(x)] + S\big(\frac{t^\alpha}{\alpha}\big)[h(x_n) - h(x)] \\ &+ \int_0^t s^{\alpha-1} S\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)[f(s, x_n(s)) - f(s, x(s)) + k(s, x_n(s)) - k(s, x(s))]ds.\end{aligned}$$

Then, by using a simple calculation, we obtain

$$\begin{aligned}|\, \Gamma(x_n) - \Gamma(x)\, |_c &\leq \sup_{t\in[0,\tau]} |C\big(\frac{t^\alpha}{\alpha}\big)| \parallel g(x_n) - g(x) \parallel + \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \parallel h(x_n) - h(x) \parallel \\ &+ \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \int_0^\tau s^{\alpha-1} \parallel [f(s, x_n(s)) - f(s, x(s)) + k(s, x_n(s)) - k(s, x(s))] \parallel ds.\end{aligned}$$

Using assumption (H_1) and (H_2), we get $\parallel s^{\alpha-1}[f(s, x_n(s)) - f(s, x(s)) + k(s, x_n(s)) - k(s, x(s))] \parallel \leq 2(\mu(s) + \nu(s))s^{\alpha-1}$ and $[f(s, x_n(s)) - f(s, x(s)) + k(s, x_n(s)) - k(s, x(s))] \longrightarrow 0$ as $n \longrightarrow +\infty$. The Lebesgue dominated convergence theorem proves that $\int_0^\tau s^{\alpha-1} \parallel [f(s, x_n(s)) - f(s, x(s)) + k(s, x_n(s)) - k(s, x(s))] \parallel ds \longrightarrow 0$ as $n \longrightarrow +\infty$. Since the continuity of the functions g and h, then we have $\lim\limits_{n\longrightarrow+\infty} \parallel g(x_n) - g(x) \parallel = 0$ and $\lim\limits_{n\longrightarrow+\infty} \parallel h(x_n) - h(x) \parallel = 0$. Hence, Γ is a continuous operator.

Step 2: Prove that Γ is compact.

`Claim 1`: Prove that $\Gamma(B_r)$ is bounded.
For $x \in B_r$ and $t \in [0, \tau]$, we have

$$\begin{aligned}|\, \Gamma(x)\, |_c &\leq \sup_{t\in[0,\tau]} |C\big(\frac{t^\alpha}{\alpha}\big)|[\parallel x_0 \parallel + \parallel g(x) \parallel] \\ &+ \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \Big[\parallel x_1 \parallel + \parallel h(x) \parallel + \int_0^\tau s^{\alpha-1} \parallel f(s, x(s)) + k(s, x(s)) \parallel ds \Big].\end{aligned}$$

Using assumptions (H_1), (H_2), (H_4) and (H_6), we get

$$\begin{aligned}|\, \Gamma(x)\, |_c &\leq \sup_{t\in[0,\tau]} |C\big(\frac{t^\alpha}{\alpha}\big)| \Big[\parallel x_0 \parallel + ar + b \Big] + \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \Big[\parallel x_1 \parallel + cr + d \Big] \\ &+ \frac{\tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \Big(|\, \mu\, |_{L^\infty([0,\tau],\, \mathbb{R}^+)} + |\, \nu\, |_{L^\infty([0,\tau],\, \mathbb{R}^+)} \Big) \\ &=: \delta.\end{aligned}$$

Hence, $\Gamma(B_r) \subseteq B_\delta$. This implies that Γ sends bounded sets into bounded sets.
`Claim 2`: Prove that $\Gamma(B_r)$ is equicontinuous.
According to Bouaouid et al. (2019), we get that the following operator

$$\Gamma_1(x)(t) = \int_0^t s^{\alpha-1} S\big(\frac{t^\alpha - s^\alpha}{\alpha}\big)[f(s, x(s)) + k(s, x(s))]ds$$

is equicontinuous. Then it is remains to show that the operator

$$\Gamma_2(x)(t) = C\big(\frac{t^\alpha}{\alpha}\big)\Big[x_0 + g(x)\Big] + S\big(\frac{t^\alpha}{\alpha}\big)\Big[x_1 + h(x)\Big]$$

is equicontinuous. To do so, let $x \in B_r$ and $t_1,\ t_2 \in [0,\tau]$ such that $t_1 < t_2$. Then, we have

$$\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) = \left[C\big(\frac{t_2^\alpha}{\alpha}\big) - C\big(\frac{t_1^\alpha}{\alpha}\big)\right]g(x) + \left[S\big(\frac{t_2^\alpha}{\alpha}\big) - S\big(\frac{t_1^\alpha}{\alpha}\big)\right]h(x).$$

This last equality combined with assumption (H_7) proves that the operator Γ_2 is equicontinuous. In consequence, the operator Γ is equicontinuous on $[0,\tau]$.

`Claim 3`: Prove that $\Gamma(B_r)(t)$ is relatively compact in X. For $t \in [0,\tau]$ let $\varepsilon \in]0,t[$, $x \in B_r$ and define the operator Γ^ε as follows

$$\Gamma^\varepsilon(x)(t) = C(\frac{t^\alpha}{\alpha})[x_0 + g(x)] + S(\frac{t^\alpha}{\alpha})[x_1 + h(x)] +$$
$$+ \int_0^{(t^\alpha - \varepsilon^\alpha)^{\frac{1}{\alpha}}} s^{\alpha-1} S(\frac{t^\alpha - s^\alpha}{\alpha})[f(s,x(s)) + k(s,x(s))]ds.$$

By using the assumptions (H_3), (H_5) and (H_8), we conclude that the set $\{\Gamma^\varepsilon(x)(t),\ x \in B_r\}$ is relatively compact in X.
By using a simple calculation combined with assumption (H_1) and (H_2), we get

$$\| \Gamma^\varepsilon(x)(t) - \Gamma(x)(t) \| \leq \left(\mid \mu \mid_{L^\infty([0,\tau],\ \mathbb{R}^+)} + \mid \nu \mid_{L^\infty([0,\tau],\ \mathbb{R}^+)} \right) \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \frac{\varepsilon^\alpha}{\alpha}.$$

Therefore, we deduce that the set $\{\Gamma(x)(t),\ x \in B_r\}$ is relatively compact in X. Finally, by using the Arzela-Ascoli theorem, we conclude that Γ is a compact operator.

Step 3: Prove that $\Gamma(\Omega)$ is bounded.

Let $x \in \Gamma(\Omega)$, then $x = \lambda\Gamma(x)$ for some $\lambda \in]0,1[$. Thus, for $t \in [0,\tau]$, we have

$$\begin{aligned}
\mid x \mid_c = \lambda \mid \Gamma(x) \mid_c &\leq \lambda \sup_{t\in[0,\tau]} |C\Big(\frac{t^\alpha}{\alpha}\Big)| \Big[\| x_0 \| + \| g(x) \| \Big] + \lambda \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \Big[\| x_1 \| + \| h(x) \| \Big] \\
&+ \lambda \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \int_0^\tau s^{\alpha-1} \| f(s,x(s)) + k(s,x(s)) \| ds \\
&\leq \lambda \sup_{t\in[0,\tau]} |C\big(\frac{t^\alpha}{\alpha}\big)| \Big[\| x_0 \| + a \mid x \mid_c + b \Big] + \lambda \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \Big[\| x_1 \| + c \mid x \mid_c + d \Big] \\
&+ \lambda \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \int_0^\tau s^{\alpha-1} \| f(s,x(s)) + k(s,x(s)) \| ds \\
&\leq \sup_{t\in[0,\tau]} |C\big(\frac{t^\alpha}{\alpha}\big)| \Big[\| x_0 \| + a \mid x \mid_c + b \Big] + \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \Big[\| x_1 \| + c \mid x \mid_c + d \Big] \\
&+ \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \frac{\tau^\alpha}{\alpha} \Big(\mid \mu \mid_{L^\infty([0,\tau],\ \mathbb{R}^+)} + \mid \nu \mid_{L^\infty([0,\tau],\ \mathbb{R}^+)} \Big).
\end{aligned}$$

Hence, one has

$$\mid x \mid_c \leq \frac{\sup_{t\in[0,\tau]} |C\big(\frac{t^\alpha}{\alpha}\big)| \Big[\| x_0 \| + b \Big] + \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)| \Big[\| x_1 \| + d + \frac{\tau^\alpha}{\alpha} \Big(\mid \mu \mid_{L^\infty([0,\tau],\ \mathbb{R}^+)} + \mid \nu \mid_{L^\infty([0,\tau],\ \mathbb{R}^+)} \Big) \Big]}{1 - a \sup_{t\in[0,\tau]} |C\big(\frac{t^\alpha}{\alpha}\big)| - c \sup_{t\in[0,\tau]} |S\big(\frac{t^\alpha}{\alpha}\big)|}.$$

This shows that the set $\Gamma(\Omega)$ is bounded.
In conclusion, by using the above steps and the Schaefer fixed point theorem, we deduce that Γ has at least one fixed point, which is a mild solution of Cauchy problem (2). $\square$

4 Conclusion

The existence of mild solutions of a Cauchy problem of nonlocal differential equations of second order with conformable fractional derivative is largely studied in the works Atraoui & Bouaouid (2021); Bouaouid et al. (2019). Our contribution in this present work is the study of mild solutions for such Cauchy problem with an integral term by means of the Schaefer fixed point theorem combined with the cosine family of linear operators.

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References

Abdeljawad, T. (2015). On conformable fractional calculus. *Journal of Computational and Applied Mathematics*, *279*, 57-66.

Atraoui, M., Bouaouid, M. (2021). On the existence of mild solutions for nonlocal differential equations of the second order with conformable fractional derivative. *Advances in Difference Equations*, *2021*, 1-11.

Bouaouid, M., Hilal, K. & Melliani, S. (2019). Sequential evolution conformable differential equations of second order with nonlocal condition. *Advances in Difference Equations*, *2019*, 21.

Bouaouid, M., Atraoui, M., Hilal, K. & Melliani, S. (2018). Fractional differential equations with nonlocal-delay condition. *Journal of Advanced Mathematical Studies*, *11*, 214-225.

Byszewski, L. (1991). Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *Journal of Mathematical Analysis and Applications*, *162*, 494-505.

Guliyev, A. (2021). On a fundamental solution of one linear ordinary differential equation with fractional order derivative. *Advanced Mathematical Models & Applications*, *6*(2), 162-173.

Khalil, R., Al Horani, M., Yousef, A. & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics, 264*, 65-70.

Laghrib, A. (2020). A comparative study between tv, tv2, btv and combined models for the multi-frame super-resolution. *Advanced Mathematical Models & Applications*, *5*(1), 80-94.

Nachaoui, M. (2020). Parameter learning for combined first and second order total variation for image reconstruction. *Advanced Mathematical Models & Applications, 5*(1), 53-69.

Nachaoui, A., Nachaoui, M. & Gasimov, B. (2021). Parallel numerical computation of an analytical method for solving an inverse problem. *Advanced Mathematical Models & Applications*, 6(2), 218-226.

Travis, C.C., Webb, G.F. (1978). Cosine family and abstract nonlinear second order differential equations. *Acta Mathematica Academiae Seientiarum Hungaricae Tomus*, *32*, 75-96.